A Proof of the Cameron-Ku Conjecture

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Abstract

A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some i such that $\sigma(i) = \pi(i)$. Deza and Frankl [3] showed that for such a family, $|\mathcal{A}| \leq (n-1)!$. Cameron and Ku [2] showed that if equality holds then $\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j\}$ for some i and j. They conjectured a 'stability' version of this result, namely that there exists a constant $c < 1$ such that if $\mathcal{A} \subset S_n$ is an intersecting family of size at least $c(n-1)!$, then there exist i and j such that every permutation in $\mathcal{A}$ maps i to j (we call such a family 'centred'). They also made the stronger 'Hilton-Milner' type conjecture that for $n \geq 6$, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ cannot be larger than the family $\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \text{ for some } i > 2\} \cup \{(12)\}$, which has size $(1 - 1/e + o(1))(n-1)!$.

We prove the stability conjecture, and also the Hilton-Milner type conjecture for n sufficiently large. Our proof makes use of the classical representation theory of S_n. One of our key tools will be an extremal result on cross-intersecting families of permutations, namely that for $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$. This was a conjecture of Leader [11]; it was proved for n sufficiently large by Friedgut, Pilpel and the author in [4].

1 Introduction

We work on the symmetric group S_n, the group of all permutations of $\{1, 2, \ldots, n\} = [n]$. A family of permutations $\mathcal{A} \subset S_n$ is said to be *intersecting* if any two permutations in $\mathcal{A}$ agree at some point, i.e. for any $\sigma, \pi \in \mathcal{A}$, there is some $i \in [n]$ such that $\sigma(i) = \pi(i)$.

It is natural to ask: how large can an intersecting family be? The family of all permutations fixing 1 is an obvious example of a large intersecting family of permutations; it has size $(n-1)!$. More generally, for any $i, j \in [n]$, the collection of all permutations mapping i to j is clearly an intersecting

family of the same size; we call these the '*1-cosets*' of S_n, since they are the cosets of the point-stabilizers.

Deza and Frankl [3] showed that if $\mathcal{A} \subset S_n$ is intersecting, then $|\mathcal{A}| \leq (n-1)!$; this is known as the Deza-Frankl Theorem. They gave a short, direct Katona-type proof (analogous to Katona's proof the the Erdős-Ko-Rado theorem on intersecting families of r-sets): take any n-cycle ρ, and let H be the cyclic group of order n generated by ρ. For any left coset σH of H, any two distinct permutations in σH disagree at every point, and therefore σH contains at most 1 member of $\mathcal{A}$. Since the left cosets of H partition S_n, it follows that $|\mathcal{A}| \leq (n-1)!$.

Deza and Frankl conjectured that equality holds only for the 1-cosets of S_n. This turned out to be much harder than expected; it was eventually proved by Cameron and Ku [2]; Larose and Malvenuto [10] independently found a different proof. One may compare the situation to that for intersecting families of r-sets of $[n]$. We say a family $\mathcal{A}$ of r-element subsets of $[n]$ is *intersecting* if any two of its sets have nonempty intersection. The classical Erdős-Ko-Rado Theorem states that for $r < n/2$, the largest intersecting families of r-sets of $[n]$ are the 'stars', i.e. the families of the form $\{x \in [n]^{(r)} : \; i \in x\}$ for $i \in [n]$.

We say that an intersecting family $\mathcal{A} \subset S_n$ is *centred* if there exist $i, j \in [n]$ such that every permutation in $\mathcal{A}$ maps i to j, i.e. $\mathcal{A}$ is contained within a 1-coset of S_n. Cameron and Ku asked how large a *non-centred* intersecting family can be. Experimentation suggests that the further an intersecting family is from being centred, the smaller it must be. The following are natural candidates for large non-centred intersecting families:

- $\mathcal{B} = \{\sigma \in S_n : \sigma$ fixes at least two points in $[3]\}$.

  This has size $3(n-2)! - 2(n-3)!$.
  It requires the removal of $(n-2)! - (n-3)!$ permutations to make it centred.

- $\mathcal{C} = \{\sigma : \sigma(1) = 1, \; \sigma$ intersects $(1\ 2)\} \cup \{(1\ 2)\}$.

  Claim: $|\mathcal{C}| = (1 - 1/e + o(1))(n-1)!$
  Proof of Claim: Let $\mathcal{D}_n = \{\sigma \in S_n : \; \sigma(i) \neq i \; \forall i \in [n]\}$ be the set of *derangements* of $[n]$ (permutations without fixed points); let $d_n = |\mathcal{D}_n|$ be the number of derangements of $[n]$. By the inclusion-exclusion for-

mula,

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = n!(1/e + o(1))$$

Note that a permutation which fixes 1 intersects (1 2) iff it has a fixed point greater than 2. The number of permutations fixing 1 alone is clearly d_{n-1}; the number of permutations fixing 1 and 2 alone is clearly d_{n-2}, so the number of permutations fixing 1 and some other point > 2 is $(n-1)! - d_{n-1} - d_{n-2}$. Hence,

$$|\mathcal{C}| = (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!$$

as required.

Note that $\mathcal{C}$ can be made centred just by removing (1 2).

For $n \leq 5$, $\mathcal{B}$ and $\mathcal{C}$ have the same size; for $n \geq 6$, $\mathcal{C}$ is larger. Cameron and Ku [2] conjectured that for $n \geq 6$, $\mathcal{C}$ has the largest possible size of any non-centred intersecting family. Further, they conjectured that any non-centred intersecting family $\mathcal{A}$ of the same size as $\mathcal{C}$ is a 'double translate' of $\mathcal{C}$, meaning that there exist $\pi, \tau \in S_n$ such that $\mathcal{A} = \pi\mathcal{C}\tau$. Note that if $\mathcal{F} \subset S_n$, any double translate of $\mathcal{F}$ has the same size as $\mathcal{F}$, is intersecting iff $\mathcal{F}$ is and is centred iff $\mathcal{F}$ is; this will be our notion of 'isomorphism' for intersecting families of permutations.

One may compare the Cameron-Ku conjecture to the Hilton-Milner theorem on intersecting families of r-sets (see [6]). We say that a family $\mathcal{A}$ of r-sets of $[n]$ is *trivial* if there is an element in all of its sets. Hilton and Milner proved that for $r \geq 4$ and $n > 2r$, if $\mathcal{A} \subset [n]^{(r)}$ is a non-trivial intersecting family of maximum size, then

$$\mathcal{A} = \{x \in [n]^{(r)} : i \in [n],\ x \cap y \neq \emptyset\} \cup \{y\}$$

for some $i \in [n]$ and some r-set y not containing i, so it can be made into a trivial family by removing just one r-set.

We prove the Cameron-Ku conjecture for n sufficiently large. This implies the weaker 'stability' conjecture of Cameron and Ku [2] that there exists a constant $c > 0$ such that any intersecting family $\mathcal{A} \subset S_n$ of size at least $(1-c)(n-1)!$ is centred. We prove the latter using a slightly shorter argument.

Our proof makes use of the classical representation theory of S_n. One of our key tools will be an extremal result for cross-intersecting families of permutations. A pair of families of permutations $\mathcal{A}, \mathcal{B} \subset S_n$ is said to be *cross-intersecting* if for any $\sigma \in \mathcal{A}, \tau \in \mathcal{B}$, σ and τ agree at some point, i.e. there is some $i \in [n]$ such that $\sigma(i) = \tau(i)$. Leader [11] conjectured that for $n \geq 4$, for such a pair, $|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$, with equality iff $\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\}$ for some $i, j \in [n]$. (Note that the statement does not hold for $n = 3$, as the pair $\mathcal{A} = \{(1), (123), (321)\},\ \mathcal{B} = \{(12), (23), (31)\}$ is cross-intersecting.)

A k-cross-intersecting generalization of Leader's conjecture was proved by Friedgut, Pilpel and the author in [4], for n sufficiently large depending on k. In order to prove the Cameron-Ku conjecture for n sufficiently large, we could in fact make do with the $k = 1$ case of this result. For completeness, however, we sketch a simpler proof of Leader's conjecture for all $n \geq 4$, based on the eigenvalues of the derangement graph rather than those of the weighted graph constructed in [4].

2 Cross-intersecting families and the derangement graph

Consider the *derangement graph* Γ on S_n, in which we join two permutations iff they disagree at every point, i.e. we join σ and τ iff $\sigma(i) \neq \tau(i)$ for every $i \in [n]$. (Γ is the Cayley graph on S_n generated by the set $\mathcal{D}_n$ of derangements, so is d_n-regular.) A cross-intersecting pair of families of permutations is simply two vertex sets $\mathcal{A}, \mathcal{B}$ with no edges of Γ between them. We will apply the following general result (of which a variant can be found in [1]) to the derangement graph:

Theorem 2.1. *(i) Let Γ be a d-regular graph on N vertices, whose adjacency matrix A has eigenvalues $\lambda_1 = d \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let $\nu = \max(|\lambda_2|, |\lambda_N|)$. Suppose X and Y are sets of vertices of Γ with no edges between them, i.e. $xy \notin E(\Gamma)$ for every $x \in X$ and $y \in Y$. Then*

$$\sqrt{|X||Y|} \leq \frac{\nu}{d+\nu} N \tag{1}$$

(ii) Suppose further that $|\lambda_2| \neq |\lambda_N|$, and let λ' be the larger in modulus of the two. Let v_X, v_Y be the characteristic vectors of X, Y and let $\mathbf{f}$ denote the all-1's vector in $\mathbb{C}^N$; if we have equality in (1), then $|X| = |Y|$, and the characteristic vectors $v_X, v_Y \in Span\{\mathbf{f}\} \oplus E(\lambda')$, the direct sum of the

d- and λ'-eigenspaces of A, or equivalently, the shifted characteristic vectors $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ are eigenvectors of A with eigenvalue λ'.

Proof. Equip $\mathbb{C}^N$ with the inner product:

$$\langle x, y\rangle = \frac{1}{N}\sum_{i=1}^{N} \bar{x}_i y_i$$

and let

$$||x|| = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |x_i|^2}$$

be the induced norm. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an orthonormal basis of real eigenvectors of A corresponding to the eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_N$. Let X, Y be as above; write

$$v_X = \sum_{i=1}^{N} \xi_i u_i, \quad v_Y = \sum_{i=1}^{N} \eta_i u_i$$

as linear combinations of the eigenvectors of A. We have $\xi_1 = \alpha$, $\eta_1 = \beta$,

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha, \quad \sum_{i=1}^{N} \eta_i^2 = ||v_Y||^2 = |Y|/N = \beta$$

Since there is no edge of Γ between X and Y, we have the crucial property:

$$0 = \sum_{x\in X, y\in Y} A_{x,y} = v_Y^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i \eta_i = d\alpha\beta + \sum_{i=2}^{N} \lambda_i \xi_i \eta_i \geq d\alpha\beta - \nu \left|\sum_{i=2}^{N} \xi_i \eta_i\right| \tag{2}$$

Provided $|\lambda_2| \neq |\lambda_N|$, if we have equality above, then $\xi_i = \eta_i = 0$ unless $\lambda_i = d$ or λ', so $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ are λ'-eigenvectors, so $v_X, v_Y \in \mathrm{Span}\{\mathbf{f}\} \oplus E(\lambda')$.

The Cauchy-Schwarz inequality gives:

$$\left|\sum_{i=2}^{N} \xi_i \eta_i\right| \leq \sqrt{\sum_{i=2}^{N} \xi_i^2 \sum_{i=2}^{N} \eta_i^2} = \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$$

Substituting this into (2) gives:

$$d\alpha\beta \leq \nu\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$$

so

$$\frac{\alpha\beta}{(1-\alpha)(1-\beta)} \leq (\nu/d)^2$$

By the AM/GM inequality, $(\alpha+\beta)/2 \geq \sqrt{\alpha\beta}$ with equality iff $\alpha = \beta$, so

$$\frac{\alpha\beta}{(1-\sqrt{\alpha\beta})^2} = \frac{\alpha\beta}{1-2\sqrt{\alpha\beta}+\alpha\beta} \leq \frac{\alpha\beta}{1-\alpha-\beta+\alpha\beta} \leq (\nu/d)^2$$

implying that

$$\sqrt{\alpha\beta} \leq \frac{\nu}{d+\nu}$$

Hence, we have

$$\sqrt{|X||Y|} \leq \frac{\nu}{d+\nu}N$$

and provided $|\lambda_2| \neq |\lambda_N|$, we have equality only if $|X| = |Y| = \frac{\nu}{d+\nu}N$ and $v_X - (|X|/N)\mathbf{f}, v_Y - (|Y|/N)\mathbf{f}$ are eigenvectors of A with eigenvalue λ', as required. $\square$

We will show that for $n \geq 5$, the derangement graph satisfies the hypotheses of this result with $\nu = d_n/(n-1)$; in fact, $\lambda_N = -\frac{d_n}{n-1}$ and all other eigenvalues are $O((n-2)!)$. Note that the eigenvalues of the derangement graph (focussing on the least eigenvalue) have been investigated by Renteln [13], Ku and Wales [9], and Godsil and Meagher [5]. The difference between our approach and theirs is that we employ a short-cut (Lemma 2.4) to bound all eigenvalues of high multiplicity. We also believe that our presentation is natural from an algebraic viewpoint.

If G is a finite group and Γ is a graph on G, the adjacency matrix A of G is a linear operator on $\mathbb{C}[G]$, the vector space of all complex-valued functions on G. Recall the following

Definition. *For a finite group G, the* group module $\mathbb{C}G$ *is the complex vector space with basis G and multiplication defined by extending the group multiplication linearly; explicitly,*

$$\left(\sum_{g\in G} x_g g\right)\left(\sum_{h\in G} y_h h\right) = \sum_{g,h\in G} x_g y_h (gh)$$

Identifying a function $f : G \to \mathbb{C}$ with $\sum_{g\in G} f(g)g$, we may consider $\mathbb{C}[G]$ as the group module $\mathbb{C}G$. If Γ is a Cayley graph on G with (inverse-closed) generating set X, the adjacency matrix of Γ acts on the group module $\mathbb{C}G$ by left multiplication by $\sum_{g\in X} g$.

We say that Γ is a normal Cayley graph if its generating set is a union of conjugacy-classes of G. The set of derangements is a union of conjugacy classes of S_n, so the derangement graph is a normal Cayley graph. The following result gives an explicit 1-1 correspondence between the (isomorphism classes of) irreducible representations of G and the eigenvalues of Γ:

Theorem 2.2. *(Frobenius-Schur-others) Let G be a finite group; let $X \subset G$ be an inverse-closed, conjugation-invariant subset of G and let Γ be the Cayley graph on G with generating set X. Let $(\rho_1, V_1), \ldots, (\rho_k, V_k)$ be a complete set of non-isomorphic irreducible representations of G — i.e., containing one representative from each isomorphism class of irreducible representations of G. Let U_i be the sum of all submodules of the group module $\mathbb{C}G$ which are isomorphic to V_i. We have*

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i$$

and each U_i is an eigenspace of A with dimension $\dim(V_i)^2$ and eigenvalue

$$\lambda_{V_i} = \frac{1}{\dim(V_i)} \sum_{g \in X} \chi_i(g)$$

where $\chi_i(g) = Trace(\rho_i(g))$ denotes the character of the irreducible representation (ρ_i, V_i).

Given $x \in \mathbb{C}G$, its projection onto the eigenspace U_i can be found as follows. Write $\mathrm{Id} = \sum_{i=1}^{k} e_i$ where $e_i \in U_i$ for each $i \in [k]$. The e_i's are called the *primitive central idempotents* of $\mathbb{C}G$; U_i is the two-sided ideal of $\mathbb{C}G$ generated by e_i, and e_i is given by the following formula:

$$e_i = \frac{\dim(V_i)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \tag{3}$$

For any $x \in \mathbb{C}G$, $x = \sum_{i=1}^{k} e_i x$ is the unique decomposition of x into a sum of elements of the U_i's; in other words, the projection of x onto U_i is $e_i x$.

Background on the representation theory of the symmetric group

We now collect the results we need from the representation theory of S_n; as in [4], our treatment follows [14] and [7]. Readers who are familiar with the representation theory of S_n may wish to skip this section.

A *partition* of n is a non-increasing sequence of positive integers summing to n, i.e. a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 1$ and $\sum_{i=1}^{k} \alpha_i = n$; we write $\alpha \vdash n$. For example, $(3,2,2) \vdash 7$; we sometimes use the shorthand $(3,2,2) = (3,2^2)$.

The *cycle-type* of a permutation $\sigma \in S_n$ is the partition of n obtained by expressing σ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}.$$

Moreover, there is an explicit 1-1 correspondence between irreducible representations of S_n (up to isomorphism) and partitions of n, which we now describe.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partiton of n. The *Young diagram* of α is an array of n dots, or cells, having k left-justified rows where row i contains α_i dots. For example, the Young diagram of the partition $(3,2^2)$ is

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}$$

If the array contains the numbers $\{1,2,\ldots,n\}$ in some order in place of the dots, we call it an α-*tableau*; for example,

$$\begin{array}{lll} 6 & 1 & 7 \\ 5 & 4 & \\ 3 & 2 & \end{array}$$

is a $(3,2^2)$-tableau. Two α-tableaux are said to be *row-equivalent* if for each row, they have the same numbers in that row. If an α-tableau t has rows $R_1, \ldots, R_k \subset [n]$ and columns $C_1, \ldots, C_l \subset [n]$, we let $R_t = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_k}$ be the row-stablizer of t and $C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_l}$ be the column-stabilizer.

An α-*tabloid* is an α-tableau with unordered row entries (or formally, a row-equivalence class of α-tableaux); given a tableau t, we write $[t]$ for the tabloid it produces. For example, the $(3,2^2)$-tableau above produces the following $(3,2^2)$-tabloid

$$\begin{array}{lll} \{1 & 6 & 7\} \\ \{4 & 5\} & \\ \{2 & 3\} & \end{array}$$

Consider the natural left action of S_n on the set X^α of all α-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, i.e. the complex vector space with basis X^α and S_n action given by extending this action linearly. Given an α-tableau t, we define the corresponding *α-polytabloid*

$$e_t := \sum_{\pi \in C_t} \epsilon(\pi)\pi[t]$$

We define the *Specht module* S^α to be the submodule of M^α spanned by the α-polytabloids:

$$S^\alpha = \mathrm{Span}\{e_t : \ t \text{ is an } \alpha\text{-tableau}\}.$$

A central observation in the representation theory of S_n is that *the Specht modules are a complete set of pairwise non-isomorphic, irreducible representations of S_n*. Hence, any irreducible representation ρ of S_n is isomorphic to some S^α. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation S.

We say that a tableau is *standard* if the numbers strictly increase along each row and down each column. It turns out that for any partition α of n,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module S^α.

Given a partition α of n, for each cell (i,j) in its Young diagram, we define the 'hook-length' ($h^\alpha_{i,j}$) to be the number of cells in its 'hook' (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook-lengths of $(3,2^2)$ are as follows:

5	4	1
3	2	
2	1	

The dimension f^α of the Specht module S^α is given by the following formula

$$f^\alpha = n!/\prod(\text{hook lengths of } [\alpha]) \tag{4}$$

From now on we will write $[\alpha]$ for the equivalence class of the irreducible representation S^α, χ_α for the irreducible character χ_{S^α}, and ξ_α for the character of the permutation representation M^α. Notice that the set of α-tabloids form a basis for M^α, and therefore $\xi_\alpha(\sigma)$, the trace of the

corresponding permutation representation at σ, is precisely the number of α-tabloids fixed by σ.

If $U \in [\alpha]$, $V \in [\beta]$, we define $[\alpha]+[\beta]$ to be the equivalence class of $U \oplus V$, and $[\alpha] \otimes [\beta]$ to be the equivalence class of $U \otimes V$; since $\chi_{U \oplus V} = \chi_U + \chi_V$ and $\chi_{U \otimes V} = \chi_U \cdot \chi_V$, this corresponds to pointwise addition/multiplication of the corresponding characters.

The Branching Theorem (see [9] §2.4) states that for any partition α of n, the restriction $[\alpha] \downarrow S_{n-1}$ is isomorphic to a direct sum of those irreducible representations $[\beta]$ of S_{n-1} such that the Young diagram of β can be obtained from that of α by deleting a single dot, i.e., if α^{i-} is the partition whose Young diagram is obtained by deleting the dot at the end of the ith row of that of α, then

$$[\alpha] \downarrow S_{n-1} = \sum_{i: \alpha_i > \alpha_{i-1}} [\alpha^{i-}] \tag{5}$$

For example, if $\alpha = (3, 2^2)$, we obtain

$$[3, 2^2] \downarrow S_6 = \left[\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}\right] + \left[\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}\right] = [2^3] + [3, 2, 1]$$

For any partition α of n, we have $S^{(1^n)} \otimes S^{\alpha} \cong S^{\alpha'}$, where α' is the transpose of α, the partition of n with Young diagram obtained by interchanging rows with columns in the Young diagram of α. Hence, $[1^n] \otimes [\alpha] = [\alpha']$, and $\chi_{\alpha'} = \epsilon \cdot \chi_{\alpha}$. For example, we obtain:

$$[3, 2, 2] \otimes [1^7] = [3, 2, 2]' = \left[\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}\right]' = \left[\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{array}\right] = [3, 3, 1]$$

We now explain how the permutation modules M^{β} decompose into irreducibles.

Definition. *Let α, β be partitions of n. A* generalized α-tableau *is produced by replacing each dot in the Young diagram of α with a number between 1 and n; if a generalized α-tableau has β_i i's $(1 \leq i \leq n)$ it is said to have* content β. *A generalized α-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

Definition. *Let α, β be partitions of n. The* Kostka number $K_{\alpha,\beta}$ *is the number of semistandard generalized α-tableaux with content β.*

Young's Rule states that for any partition β of n, the permutation module M^β decomposes into irreducibles as follows:

$$M^\beta \cong \oplus_{\alpha \vdash n} K_{\alpha,\beta} S^\alpha$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of S_n on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)}$$

and therefore

$$\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1 \tag{6}$$

We now return to considering the derangement graph. Write U_α for the sum of all copies of S^α in $\mathbb{C}S_n$. Note that $U_{(n)} = \mathrm{Span}\{\boldsymbol{f}\}$ is the subspace of constant vectors in $\mathbb{C}S_n$. Applying Theorem 2.2 to the derangement graph Γ, we have

$$\mathbb{C}S_n = \bigoplus_{\alpha \vdash n} U_\alpha$$

and each U_α is an eigenspace of the derangement graph, with dimension $\dim(U_\alpha) = (f^\alpha)^2$ and corresponding eigenvalue

$$\lambda_\alpha = \frac{1}{f^\alpha} \sum_{\sigma \in \mathcal{D}_n} \chi_\alpha(\sigma) \tag{7}$$

We will use the following result, a variant of which is proved in [7]; for the reader's convenience, we include a proof using the Branching Theorem and the Hook Formula.

Lemma 2.3. *For $n \geq 9$, the only Specht modules S^α of dimension $f^\alpha < \binom{n-1}{2} - 1$ are as follows:*

- $S^{(n)}$ *(the trivial representation), dimension 1*
- $S^{(1^n)}$ *(the sign representation S), dimension 1*
- $S^{(n-1,1)}$*, dimension $n-1$*
- $S^{(2,1^{n-2})}$ *($\cong S \otimes S^{(n-1,1)}$), dimension $n-1$*

$$(*)$$

This is well-known, but for completeness we include a proof using the Branching Theorem and the Hook Formula.

Proof. By direct calculation using (4) the lemma can be verified for $n = 9, 10$. We proceed by induction. Assume the lemma holds for $n-2, n-1$; we will prove it for n. Let α be a partition of n such that $f^\alpha < \binom{n-1}{2} - 1$. Consider the restriction $[\alpha] \downarrow S_{n-1}$, which has the same dimension. First suppose $[\alpha] \downarrow S_{n-1}$ is reducible. If it has one of our 4 irreducible representations $(*)$ as a constituent, then by (5), the possibilies for α are as follows:

constituent	possibilies for α
$[n-1]$	$(n), (n-1, 1)$
$[1^{n-1}]$	$(1^n), (2, 1^{n-1})$
$[n-2, 1]$	$(n-1, 1), (n-2, 2), (n-2, 1, 1)$
$[2, 1^{n-3}]$	$(2, 1^{n-2}), (2, 2, 1^{n-4}), (3, 1^{n-3})$

But using (4), the new irreducible representations above all have dimension $\geq \binom{n-1}{2} - 1$:

α	f^α
$(n-2, 2), (2, 2, 1^{n-4})$	$\binom{n-1}{2} - 1$
$(n-2, 1, 1), (3, 1^{n-3})$	$\binom{n-1}{2}$

hence none of these are constituents of $[\alpha] \downarrow S_{n-1}$. So WMA the irreducible constituents of $[\alpha] \downarrow S_{n-1}$ don't include any of our 4 irreducible representations $(*)$, hence by the induction hypothesis for $n-1$, each has dimension $\geq \binom{n-2}{2} - 1$. But $2(\binom{n-2}{2} - 1) \geq \binom{n-1}{2} - 1$ provided $n \geq 11$, hence there is just one, i.e. $[\alpha] \downarrow S_{n-1}$ is irreducible. Therefore $[\alpha] = [s^t]$ for some $s, t \in \mathbb{N}$ with $st = n$, i.e. it has square Young diagram. Now consider

$$[\alpha] \downarrow S_{n-2} = [s^{t-1}, s-2] + [s^{t-2}, s-1, s-1]$$

Note that neither of these 2 irreducible constituents are any of our 4 irreducible representations $(*)$, hence by the induction hypothesis for $n-2$, each has dimension $\geq \binom{n-3}{2} - 1$, but $2(\binom{n-3}{2} - 1) \geq \binom{n-1}{2} - 1$ for $n \geq 11$, contradicting $\dim([\alpha] \downarrow S_{n-2}) < \binom{n-1}{2} - 1$. $\square$

If α is any partition of n whose Specht module has high dimension $f^\alpha \geq \binom{n-1}{2} - 1$, we may bound $|\lambda_\alpha|$ using the following trick:

Lemma 2.4. *Let Γ be a graph on N vertices whose adjancency matrix A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$; then*

$$\sum_{i=1}^{N} \lambda_i^2 = 2e(\Gamma)$$

This is well-known; we include a proof for completeness.

Proof. Diagonalize A: there exists a real invertible matrix P such that $A = P^{-1}DP$, where D is the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_N \end{pmatrix}$$

We have $A^2 = P^{-1}D^2P$, and therefore

$$2e(\Gamma) = \sum_{i,j=1}^{N} A_{i,j} = \sum_{i,j=1}^{N} A_{i,j}^2 = \mathrm{Tr}(A^2) = \mathrm{Tr}(P^{-1}D^2P) = \mathrm{Tr}(D^2) = \sum_{i=1}^{N} \lambda_i^2$$

as required. $\square$

Hence, the eigenvalues of the derangement graph satisfy:

$$\sum_{\alpha \vdash n} (f^{\alpha}\lambda_{\alpha})^2 = 2e(\Gamma) = n!d_n = (n!)^2(1/e + o(1))$$

so for each partition α of n,

$$|\lambda_{\alpha}| \leq \frac{\sqrt{n!d_n}}{f^{\alpha}} = \frac{n!}{f^{\alpha}}\sqrt{1/e + o(1)}$$

Therefore, if S^{α} has dimension $f^{\alpha} \geq \binom{n-1}{2} - 1$, then $|\lambda_{\alpha}| \leq O((n-2)!)$. For each of the Specht modules $(*)$, we now explicitly calculate the corresponding eigenvalue using (7).

For the trivial module, $\chi_{(n)} \equiv 1$, so

$$\lambda_{(n)} = d_n$$

For the sign module $S^{(1^n)}$, $\chi_{(1^n)} = \epsilon$ so

$$\lambda_{(1^n)} = \sum_{\sigma \in \mathcal{D}_n} \epsilon(\sigma) = e_n - o_n$$

where e_n, o_n are the number of even and odd derangements of $[n]$, respectively. It is well known that for any $n \in \mathbb{N}$,

$$e_n - o_n = (-1)^{n-1}(n-1) \tag{8}$$

To see this, note that an odd permutation $\sigma \in S_n$ without fixed points can be written as $(i\ n)\rho$, where $\sigma(n) = i$, and ρ is either an even permutation of $[n-1] \setminus \{i\}$ with no fixed points (if $\sigma(i) = n$), or an even permutation of $[n-1]$ with no fixed points (if $\sigma(i) \neq n$). Conversely, for any $i \neq n$, if ρ is any even permutation of $[n-1]$ with no fixed points or any even permutation of $[n-1] \setminus \{i\}$ with no fixed points, then $(i\ n)\rho$ is a permutation of $[n]$ with no fixed points taking $n \mapsto i$. Hence, for all $n \geq 3$,

$$o_n = (n-1)(e_{n-1} + e_{n-2})$$

Similarly,

$$e_n = (n-1)(o_{n-1} + o_{n-2})$$

(8) follows by induction on n.

Hence, we have:

$$\lambda_{(1^n)} = (-1)^{n-1}(n-1)$$

For the partition $(n-1, 1)$, from (6) we have:

$$\chi_{(n-1,1)}(\sigma) = \xi_{(n-1,1)}(\sigma) - 1 = \#\{\text{fixed points of } \sigma\} - 1$$

so we get

$$\lambda_{(n-1,1)} = \frac{1}{n-1} \sum_{\sigma \in \mathcal{D}_n} (-1) = -\frac{d_n}{n-1}$$

For $S^{(2,1^{n-2})} \cong S^{(1^n)} \otimes S^{(n-1,1)}$, $\chi_{(2,1^{n-2})} = \epsilon \cdot \chi_{(n-1,1)}$, so

$$\chi_{(2,1^{n-2})}(\sigma) = \epsilon(\sigma)(\#\{\text{fixed points of } \sigma\} - 1)$$

and therefore

$$\lambda_{(2,1^{n-2})} = \frac{1}{n-1} \sum_{\sigma \in \mathcal{D}_n} -\epsilon(\sigma) = -\frac{e_n - o_n}{n-1} = (-1)^n$$

To summarize, we obtain:

α	λ_α
(n)	d_n
(1^n)	$(-1)^{n-1}(n-1)$
$(n-1,1)$	$-d_n/(n-1)$
$(2,1^{n-2})$	$(-1)^n$

Hence, $U_{(n)}$ is the d_n-eigenspace, $U_{(n-1,1)}$ is the $-d_n/(n-1)$-eigenspace, and all other eigenvalues are $O((n-2)!)$. Hence, Leader's conjecture follows (for n sufficiently large) by applying Theorem 2.1 to the derangement graph. It is easy to check that $\nu = d_n/(n-1)$ for all $n \geq 4$, giving

Theorem 2.5. *If $n \geq 4$, then any cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subset S_n$ satisfy*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)!)^2$$

If equality holds, then by Theorem 2.1 part *(ii)*, the characteristic vectors $v_{\mathcal{A}}, v_{\mathcal{B}}$ must lie in the direct sum of the d_n and $-d_n/(n-1)$-eigenspaces. It can be checked that for $n \geq 5$, $|\lambda_\alpha| < d_n/(n-1)\ \forall \alpha \neq (n), (n-1,1)$, so the d_n eigenspace is precisely $U_{(n)}$ and the $-d/(n-1)$-eigenspace is precisely $U_{(n-1,1)}$. But we have:

Lemma 2.6. *For $i, j \in [n]$, let $v_{i \mapsto j} = v_{\{\sigma \in S_n:\ \sigma(i)=j\}}$ be the characteristic vector of the 1-coset $\{\sigma \in S_n :\ \sigma(i) = j\}$. Then*

$$U_{(n)} \oplus U_{(n-1,1)} = Span\{v_{i \mapsto j}\ : i, j \in [n]\}$$

This is a special case of a theorem in [4]. We give a short proof for completeness.

Proof. Let

$$U = \mathrm{Span}\{v_{i \mapsto j}\ : i, j \in [n]\}$$

For each $i \in [n]$, $\{v_{i,j}\ : j \in [n]\}$ is a basis for a copy W_i of the permutation module $M^{(n-1,1)}$ in $\mathbb{C}S_n$. Since

$$M^{(n-1,1)} \cong S^{(n)} \oplus S^{(n-1,1)}$$

we have the decomposition

$$W_i = \mathrm{Span}\{\boldsymbol{f}\} \oplus V_i$$

where V_i is some copy of $S^{(n-1,1)}$ in $\mathbb{C}S_n$, so

$$\mathrm{Span}\{v_{i \mapsto j}\ : j \in [n]\} = W_i \leq U_{(n)} \oplus U_{(n-1,1)}$$

for each $i \in [n]$, and therefore $U \leq U_{(n)} \oplus U_{(n-1,1)}$.

It is well known that if G is any finite group, and T, T' are two isomorphic submodules of $\mathbb{C}G$, then there exists $s \in \mathbb{C}G$ such that the right multiplication map $x \mapsto xs$ is an isomorphism from T to T' (see for example [8]). Hence, for any $i \in [n]$, the sum of all right translates of W_i contains $\mathrm{Span}\{\boldsymbol{f}\}$ and all submodules of $\mathbb{C}S_n$ isomorphic to $S^{(n-1,1)}$, so $U_{(n)} \oplus U_{(n-1,1)} \leq U$. Hence, $U = U_{(n)} \oplus U_{(n-1,1)}$ as required. □

Hence, for $n \geq 5$, if equality holds in Theorem 2.5, then the characteristic vectors of $\mathcal{A}$ and $\mathcal{B}$ are linear combinations of the characteristic vectors of the 1-cosets. It was proved in [4] that if the characteristic vector of $\mathcal{A} \subset S_n$ is a linear combination of the characteristic vectors of the 1-cosets, then $\mathcal{A}$ is a disjoint union of 1-cosets. It follows that for $n \geq 5$, if equality holds in Theorem 2.5, then $\mathcal{A}$ and $\mathcal{B}$ are both disjoint unions of 1-cosets. Since they are cross-intersecting, they must both be equal to the same 1-coset, i.e.

$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \ \sigma(i) = j\}$$

for some $i, j \in [n]$. It is easily checked that the same conclusion holds when $n = 4$, so we have the following characterization of the case of equality in Leader's conjecture:

Theorem 2.7. *For $n \geq 4$, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families satisfying*

$$|\mathcal{A}||\mathcal{B}| = ((n-1)!)^2$$

then

$$\mathcal{A} = \mathcal{B} = \{\sigma \in S_n : \ \sigma(i) = j\}$$

for some $i, j \in [n]$.

3 Stability

We will now perform a stability analysis for intersecting families of permutations. First, we prove a 'rough' stability result: for any positive constant $c > 0$, if $\mathcal{A}$ is an intersecting family of permutations of size $|\mathcal{A}| \geq c(n-1)!$, then there exist i and j such that all but $O((n-2)!)$ permutations in $\mathcal{A}$ map i to j, i.e. $\mathcal{A}$ is 'almost' centred. In other words, writing $\mathcal{A}_{i \mapsto j}$ for the collection of all permutations in $\mathcal{A}$ mapping i to j, $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$. To prove this, we will first show that if $\mathcal{A}$ is an intersecting family of size at least $c(n-1)!$, then the characteristic vector $v_{\mathcal{A}}$ of $\mathcal{A}$ cannot be too far from the subspace U spanned by the characteristic vectors of the 1-cosets, the intersecting families of maximum size $(n-1)!$. We will use this to show that there exist $i, j \in [n]$ such that $|\mathcal{A}_{i \mapsto j}| \geq \omega((n-2)!)$. Clearly, for any fixed $i \in [n]$,

$$\sum_{j=1}^{n} |\mathcal{A}_{i \mapsto j}| = |\mathcal{A}|$$

and therefore the average size of an $|\mathcal{A}_{i \mapsto k}|$ is $|\mathcal{A}|/n$; $|\mathcal{A}_{i \mapsto j}|$ is ω of the average size. This statement would at first seem too weak to help us, but

combining it with the fact that $\mathcal{A}$ is intersecting, we may 'boost' it to the much stronger statement $|\mathcal{A}_{i\mapsto j}| \geq (1-o(1))|\mathcal{A}|$. In detail, we will deduce from Theorem 2.5 that for any $j \neq k$,

$$|\mathcal{A}_{i\mapsto j}||\mathcal{A}_{i\mapsto k}| \leq ((n-2)!)^2$$

giving $|\mathcal{A}_{i\mapsto k}| \leq o((n-2)!)$ for any $k \neq j$. Summing over all $k \neq j$ will give $|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq o((n-1)!)$, enabling us to complete the proof.

Note that this is enough to prove the stability conjecture of Cameron and Ku: if $\mathcal{A}$ is non-centred, it must contain some permutation τ such that $\tau(i) \neq j$. This immediately forces $|\mathcal{A}_{i\mapsto j}|$ to be less than $(1-1/e+o(1))(n-1)!$, yielding a contradiction if $c > 1-1/e$, and n is sufficiently large depending on c.

Here then is our rough stability result:

Theorem 3.1. *Let $c > 0$ be a positive constant. If $\mathcal{A} \subset S_n$ is an intersecting family of permutations of size $|\mathcal{A}| \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ map i to j.*

Proof. We begin with a straightforward consequence of the proof of Hoffman's theorem. Let Γ be a d-regular graph on N vertices, whose adjacency matrix A has eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let λ_M be the negative eigenvalue of second largest modulus. Let $X \subset V(\Gamma)$ be an independent set; let $\alpha = |X|/N$. Hoffman's theorem states that

$$|X| \leq \frac{|\lambda_N|}{d+|\lambda_N|}N \tag{9}$$

Let $\mathbf{f}$ be the all-1's vector in $\mathbb{C}^N$; let $U = \text{Span}\{\mathbf{f}\} \oplus E(\lambda_N)$ be the direct sum of the subspace of constant vectors and the λ_N-eigenspace of A. Let v_X be the characteristic vector of X. Hoffman's Theorem states that if equality holds in (9), then $v_X \in U$. We now derive a 'softened' version of this statement.

Equip $\mathbb{C}^N$ with the inner product

$$\langle x, y\rangle = \frac{1}{N}\sum_{i=1}^{N} \bar{x}_i y_i$$

We may bound $D = ||P_{U^\perp}(v_X)||$, the Euclidean distance from v_X to U, in terms of $|X|$, $|\lambda_N|$ and $|\lambda_M|$, as follows. Let $u_1 = \mathbf{f}, u_2, \ldots, u_N$ be an

orthonormal basis of real eigenvectors of A corresponding to the eigenvalues $\lambda_1 = d, \lambda_2, \dots, \lambda_N$. Write

$$v_X = \sum_{i=1}^{N} \xi_i u_i$$

as a linear combination of the eigenvectors of A. We have $\xi_1 = \alpha$ and

$$\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = \alpha$$

Since X is an independent set in Γ, we have the crucial property

$$0 = \sum_{x,y \in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i^2 \geq d\xi_1^2 + \lambda_N \sum_{i:\lambda_i = \lambda_N} \xi_i^2 + \lambda_M \sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2$$

Note that

$$\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2 = D^2$$

and

$$\sum_{i:\lambda_i = \lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2$$

so we have

$$0 \geq d\alpha^2 + \lambda_N(\alpha - \alpha^2 - D^2) + \lambda_M D^2$$

Rearranging, we obtain:

$$D^2 \leq \frac{(1-\alpha)|\lambda_N| - d\alpha}{|\lambda_N| - |\lambda_M|}\alpha$$

Applying this result to an independent set $\mathcal{A}$ in the derangement graph Γ, which has $|\lambda_M| \leq O((n-2)!)$, we obtain

$$\begin{aligned} D^2 &\leq \frac{(1-\alpha)d_n/(n-1) - d_n\alpha}{d_n/(n-1) - |\lambda_M|} \frac{|\mathcal{A}|}{n!} \\ &= \frac{1 - \alpha - \alpha(n-1)}{1 - (n-1)|\lambda_M|/d_n} \frac{|\mathcal{A}|}{n!} \\ &= \frac{1 - \alpha n}{1 - O(1/n)} \frac{|\mathcal{A}|}{n!} \\ &= (1 - \alpha n)(1 + O(1/n))|\mathcal{A}|/n! \end{aligned}$$

Write $|\mathcal{A}| = (1-\delta)(n-1)!$, where $\delta < 1$. Then

$$D^2 = ||P_{U^\perp}(v_\mathcal{A})||^2 \leq \delta(1+O(1/n))|\mathcal{A}|/n! \tag{10}$$

We now derive a formula for $P_U(v_A)$. The projection of $v_\mathcal{A}$ onto $U_{(n)} = \text{Span}\{\mathbf{f}\}$ is clearly $(|\mathcal{A}|/n!)\mathbf{f}$. By (3), the primitive central idempotent generating $U_{(n-1,1)}$ is

$$\frac{n-1}{n!}\sum_{\pi\in S_n}\chi_{(n-1,1)}(\pi^{-1})\pi$$

and therefore the projection of $v_\mathcal{A}$ onto $U_{(n-1,1)}$ is given by

$$P_{U_{(n-1,1)}}(v_\mathcal{A}) = \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}\sum_{\pi\in S_n}\chi_{(n-1,1)}(\pi^{-1})\pi\rho$$

which has σ-coordinate

$$\begin{aligned}
P_{U_{(n-1,1)}}(v_\mathcal{A})_\sigma &= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}\chi_{(n-1,1)}(\rho\sigma^{-1}) \\
&= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\xi_{(n-1,1)}(\rho\sigma^{-1})-1) \\
&= \frac{n-1}{n!}\sum_{\rho\in\mathcal{A}}(\#\{\text{fixed points of } \rho\sigma^{-1}\}-1) \\
&= \frac{n-1}{n!}(\#\{(\rho,i) : \rho\in\mathcal{A},\ i\in[n],\ \rho(i)=\sigma(i)\} - |\mathcal{A}|) \\
&= \frac{n-1}{n!}\sum_{i=1}^{n}|\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{n-1}{n!}|\mathcal{A}|
\end{aligned}$$

Hence, the σ-coordinate P_σ of the projection of $v_\mathcal{A}$ onto $U = U_{(n)}\oplus U_{(n-1,1)}$ is given by

$$P_\sigma = \frac{n-1}{n!}\sum_{i=1}^{n}|\mathcal{A}_{i\mapsto\sigma(i)}| - \frac{(n-2)}{n!}|\mathcal{A}|$$

which is a linear function of the number of times σ agrees with a permutation in $\mathcal{A}$.

From (10),

$$\sum_{\sigma\in\mathcal{A}}(1-P_\sigma)^2 + \sum_{\sigma\notin\mathcal{A}}P_\sigma^2 \leq |\mathcal{A}|\delta(1+O(1/n))$$

Choose $C > 0 : |\mathcal{A}|(1-1/n)\delta(1+C/n) \geq \text{RHS}$; then $(1-P_\sigma)^2 < \delta(1+C/n)$ for at least $|\mathcal{A}|/n$ permutations in $\mathcal{A}$, so the subset

$$\mathcal{A}' := \{\sigma \in \mathcal{A} : (1-P_\sigma)^2 < \delta(1+C/n)\}$$

has size at least $|\mathcal{A}|/n$. Similarly, $P_\sigma^2 < 2\delta/n$ for all but at most

$$n|\mathcal{A}|(1+O(1/n))/2 = (1-\delta)n!(1+O(1/n))/2$$

permutations $\sigma \notin \mathcal{A}$, so the subset $\mathcal{T} = \{\sigma \notin \mathcal{A} : P_\sigma^2 < 2\delta/n\}$ has size

$$|\mathcal{T}| \geq n! - (1-\delta)(n-1)! - (1-\delta)n!(1+O(1/n))/2$$

The permutations $\sigma \in \mathcal{A}'$ have P_σ close to 1; the permutations $\pi \in \mathcal{T}$ have P_π close to 0. Using only the lower bounds on the sizes of $\mathcal{A}'$ and $\mathcal{T}$, we may prove the following:

Claim: There exist permutations $\sigma \in \mathcal{A}'$, $\pi \in \mathcal{T}$ such that $\sigma^{-1}\pi$ is a product of at most $h = h(n)$ transpositions, where $h = 2\sqrt{2(n-1)\log n}$.

Proof of Claim: Define the *transposition graph* H to be the Cayley graph on S_n generated by the transpositions, i.e. $V(H) = S_n$ and $\sigma\pi \in E(H)$ iff $\sigma^{-1}\pi$ is a transposition. We use an isoperimetric inequality for H, essentially the martingale inequality of Maurey:

Theorem 3.2. *Let $X \subset V(H)$ with $|X| \geq an!$ where $0 < a < 1$. Then for any $h \geq h_0 := \sqrt{\frac{1}{2}(n-1)\log\frac{1}{a}}$,*

$$|N_h(X)| \geq \left(1 - e^{-\frac{2(h-h_0)^2}{n-1}}\right) n!$$

For a proof, see for example [12]. Applying this to the set $\mathcal{A}'$, which has $|\mathcal{A}'| \geq \frac{c(n-1)!}{n} \geq \frac{n!}{n^4}$, with $a = 1/n^4,\ h = 2h_0$, gives $|N_h(\mathcal{A}')| \geq (1-n^{-4})n!$, so certainly $N_h(\mathcal{A}') \cap \mathcal{T} \neq \emptyset$, proving the claim.

We now have two permutations $\sigma \in \mathcal{A}$, $\pi \notin \mathcal{A}$ which are 'close' to one another in H (differing in only $O(\sqrt{n\log n})$ transpositions) such that $P_\sigma > 1 - \sqrt{\delta(1+C/n)}$ and $P_\pi < \sqrt{2\delta/n}$, and therefore $P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/\sqrt{n})$, i.e. σ agrees many more times than π with permutations in $\mathcal{A}$:

$$\sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\sigma(i)}| - \sum_{i=1}^{n} |\mathcal{A}_{i\mapsto\pi(i)}| \geq (n-1)!(1-\sqrt{\delta} - O(1/\sqrt{n}))$$

Suppose for this pair we have $\pi = \sigma\tau_1\tau_2\dots\tau_l$ for transpositions $\tau_1, \dots, \tau_l$, where $l \leq t$. Let I be the set of numbers appearing in these transpositions; then $|I| \leq 2l \leq 2t$, and $\sigma(i) = \pi(i)$ for each $i \notin I$. Hence,

$$\sum_{i \in I} |\mathcal{A}_{i \mapsto \sigma(i)}| - \sum_{i \in I} |\mathcal{A}_{i \mapsto \pi(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n}))$$

so certainly,

$$\sum_{i \in I} |\mathcal{A}_{i \mapsto \sigma(i)}| \geq (n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n}))$$

By averaging,

$$\begin{aligned} |\mathcal{A}_{i \mapsto \sigma(i)}| &\geq \frac{1}{|I|}(n-1)!(1 - \sqrt{\delta} - O(1/\sqrt{n})) \\ &\geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1 - \sqrt{\delta} - O(1/\sqrt{n})) \end{aligned}$$

for some $i \in I$. Let $\sigma(i) = j$; then

$$|\mathcal{A}_{i \mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1 - \sqrt{1-c} - O(1/\sqrt{n})) = \omega((n-2)!)$$

We will now use Theorem 2.5 to show that $|\mathcal{A}_{i \mapsto k}|$ is small for each $k \neq j$. Notice that for each $k \neq j$, the pair $\mathcal{A}_{i \mapsto j}, \mathcal{A}_{i \mapsto k}$ is cross-intersecting.

Lemma 3.3. *Let $\mathcal{A} \subset S_n$ be an intersecting family; then for all i, j and k with $k \neq j$,*

$$|\mathcal{A}_{i \mapsto j}||\mathcal{A}_{i \mapsto k}| \leq ((n-2)!)^2$$

Proof. By double translation, we may assume that $i = j = 1$ and $k = 2$. Let $\sigma \in \mathcal{A}_{1 \mapsto 1}$ and $\pi \in \mathcal{A}_{1 \mapsto 2}$; then there exists $p \neq 1$ such that $\sigma(p) = \pi(p) > 2$. Hence, the translates $\mathcal{E} = \mathcal{A}_{1 \mapsto 1}$ and $\mathcal{F} = (1\ 2)\mathcal{A}_{1 \mapsto 2}$ are families of permutations fixing 1 and cross-intersecting on the domain $\{2, 3, \dots, n\}$. Deleting 1 from each permutation in the two families gives a cross-intersecting pair $\mathcal{E}', \mathcal{F}'$ of families of permutations of $\{2, 3, \dots, n\}$; applying Theorem 2.5 gives:

$$|\mathcal{A}_{1 \mapsto 1}||\mathcal{A}_{1 \mapsto 2}| = |\mathcal{E}'||\mathcal{F}'| \leq ((n-2)!)^2$$

□

Since $|\mathcal{A}_{i\mapsto j}| \geq \omega((n-2)!)$, $|\mathcal{A}_{i\mapsto k}| \leq o((n-2)!)$ for all $k \neq j$, so summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| = \sum_{k\neq j} |\mathcal{A}_{i\mapsto k}| \leq o((n-1)!)$$

and therefore

$$|\mathcal{A}_{i\mapsto j}| = |\mathcal{A}| - |\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \geq (c - o(1))(n-1)! \tag{11}$$

Applying Lemma 3.3 again gives

$$|\mathcal{A}_{i\mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$; summing over all $k \neq j$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!)$$

proving Theorem 3.1. □

The stability conjecture of Cameron and Ku follows easily.

Corollary 3.4. *Let $c > 1 - 1/e$; then for n sufficiently large depending on c, any intersecting family $\mathcal{A} \subset S_n$ of size $|\mathcal{A}| \geq c(n-1)!$ is centred.*

Proof. By Theorem 3.1, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i\mapsto j}| \geq (c - O(1/n))(n-1)! \tag{12}$$

Suppose for a contradiction that $\mathcal{A}$ is non-centred. Then there exists a permutation $\tau \in \mathcal{A}$ such that $\tau(i) \neq j$. Any permutation in $\mathcal{A}_{i\mapsto j}$ must agree with τ at some point. But for any $i, j \in [n]$ and any $\tau \in S_n$ such that $\tau(i) \neq j$, the number of permutations in S_n which map i to j and agree with τ at some point is

$$(n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e - o(1))(n-1)!$$

(By double translation, we may assume that $i = j = 1$ and $\tau = (1\ 2)$; we observed above that the number of permutations fixing 1 and intersecting $(1\ 2)$ is $(n-1)! - d_{n-1} - d_{n-2}$.) This contradicts (12) provided n is sufficiently large depending on c. □

We now use our rough stability result to prove the Hilton-Milner type conjecture of Cameron and Ku, for n sufficiently large. First, we introduce an extra notion which will be useful in the proof. Following Cameron and Ku [2], given a permutation $\pi \in S_n$ and $i \in [n]$, we define the *i-fix* of π to be the permutation π_i which fixes i, maps the preimage of i to the image of i, and agrees with π at all other points of $[n]$, i.e.

$$\pi_i(i) = i;\ \pi_i(\pi^{-1}(i)) = \pi(i);\ \pi_i(k) = \pi(k)\ \forall k \neq i, \pi^{-1}(i)$$

In other words, $\pi_i = \pi(\pi^{-1}(i)\ i)$. We inductively define

$$\pi_{i_1,\ldots,i_l} = (\pi_{i_1,\ldots,i_{l-1}})_{i_l}$$

Notice that if σ fixes j, then σ agrees with π_j wherever it agrees with π.

Theorem 3.5. *For n sufficiently large, if $\mathcal{A} \subset S_n$ is a non-centred intersecting family, then $\mathcal{A}$ is at most as large as the family*

$$\mathcal{C} = \{\sigma \in S_n : \sigma(1) = 1, \sigma(i) = i \textit{ for some } i > 2\} \cup \{(12)\}$$

which has size $(n-1)! - d_{n-1} - d_{n-2} + 1 = (1 - 1/e + o(1))(n-1)!$. Equality holds iff $\mathcal{A}$ is a double translate of $\mathcal{C}$, i.e. $\mathcal{A} = \pi\mathcal{C}\tau$ for some $\pi, \tau \in S_n$.

Proof. Let $\mathcal{A}$ be a non-centred intersecting family the same size as $\mathcal{C}$; we must show that $\mathcal{A}$ is a double translate of $\mathcal{C}$. By Theorem 3.1, there exist $i, j \in [n]$ such that $|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$, and therefore

$$|\mathcal{A}_{i \mapsto j}| \geq (n-1)! - d_{n-1} - d_{n-2} + 1 - O(n-2)! = (1 - 1/e - o(1))(n-1)!$$

Since $\mathcal{A}$ is non-centred, it must contain some permutation ρ such that $\rho(i) \neq j$. By double translation, we may assume that $i = j = 1$ and $\rho = (1\ 2)$; we will show that under these hypotheses, $\mathcal{A} = \mathcal{C}$. We have

$$|\mathcal{A}_{1 \mapsto 1}| \geq (1 - 1/e - o(1))(n-1)! \tag{13}$$

and $(1\ 2) \in \mathcal{A}$. Note that every permutation in $\mathcal{A}$ must intersect $(1\ 2)$, and therefore

$$\mathcal{A}_{1 \mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}$$

We need to show that $(1\ 2)$ is the only permutation in $\mathcal{A}$ that does not fix 1. Suppose for a contradiction that $\mathcal{A}$ contains some other permutation π not fixing 1. Then π must shift some point $p > 2$. If σ fixes both 1 and p, then σ agrees with $\pi_{1,p} = (\pi_1)_p$ wherever it agrees with π. There are exactly d_{n-2} permutations which fix 1 and p and disagree with $\pi_{1,p}$ at every point

of $\{2, \ldots, n\} \setminus \{p\}$; each disagrees everywhere with π, so none are in $\mathcal{A}$, and therefore

$$|\mathcal{A}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}$$

Hence, by assumption,

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}| \geq d_{n-2} + 1 = \Omega((n-2)!)$$

Notice that we have the following trivial bound on the size of a t-intersecting family $\mathcal{F} \subset S_n$:

$$|\mathcal{F}| \leq \binom{n}{t}(n-t)! = n!/t!$$

since every permutation in $\mathcal{F}$ must agree with a fixed $\rho \in \mathcal{F}$ in at least t places.

Hence, $\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}$ cannot be $(\log n)$-intersecting and therefore contains two permutations ρ, τ agreeing on at most $\log n$ points. The number of permutations fixing 1 and agreeing with both τ_1 and τ_2 at one of these points is at most $(\log n)(n-2)!$. All other permutations in $\mathcal{A} \cap \mathcal{C}$ agree with ρ and τ at two separate points of $\{2, \ldots, n\}$, and by the above argument, the same holds for the 1-fixes ρ_1 and τ_1. The number of permutations fixing 1 that agree with ρ_1 and τ_1 at two separate points of $\{2, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-1)!$ (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2 + o(1)$). Hence,

$$\begin{aligned}
|\mathcal{A}_{1\mapsto 1} &\leq ((1-1/e)^2 + o(1))(n-1)! + (\log n)(n-2)! \\
&= ((1-1/e)^2 + o(1))(n-1)!
\end{aligned}$$

contradicting (13) provided n is sufficiently large.

Hence, $(1\ 2)$ is the only permutation in $\mathcal{A}$ that does not fix 1, so $\mathcal{A} = \mathcal{A}_{1\mapsto 1} \cup \{(1\ 2)\} \subset \mathcal{C}$; since $|\mathcal{A}| = |\mathcal{C}|$, we have $\mathcal{A} = \mathcal{C}$ as required. □

We now perform a very similar stability analysis for cross-intersecting families. First, we prove a 'rough' stability result analogous to Theorem 3.1, namely that for any positive constant $c > 0$, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a pair of cross-intersecting families of permutations with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map i to j.

Theorem 3.6. *Let $c > 0$ be a positive constant. If $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families with $\sqrt{|\mathcal{A}||\mathcal{B}|} \geq c(n-1)!$, then there exist $i, j \in [n]$ such that all but at most $O((n-2)!)$ permutations in $\mathcal{A}$ and all but at most $O((n-2)!)$ permutations in $\mathcal{B}$ map i to j.*

Proof. Let $|\mathcal{A}| \leq |\mathcal{B}|$. First we examine the proof of Theorem 2.1 to bound $D = ||P_{U^\perp}(v_X)||,\ E = ||P_{U^\perp}(v_Y)||$. This time, we have

$$\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2 = D^2$$

$$\sum_{i>1:\lambda_i \neq \lambda_N} \eta_i^2 = E^2$$

$$\sum_{i>1:\lambda_i = \lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2$$

$$\sum_{i>1:\lambda_i = \lambda_N} \eta_i^2 = \beta - \beta^2 - E^2$$

Substituting into (2) gives:

$$\begin{aligned}
d\alpha\beta &= -\sum_{i>1:\lambda_i \neq \lambda_N} \lambda_i \xi_i \eta_i - \lambda_N \sum_{i>1:\lambda_i = \lambda_N} \xi_i \eta_i \\
&\leq \mu \sum_{i>1:\lambda_i \neq \lambda_N} |\xi_i||\eta_i| + |\lambda_N| \sum_{i>1:\lambda_i = \lambda_N} |\xi_i||\eta_i| \\
&\leq \mu \sqrt{\sum_{i>1:\lambda_i \neq \lambda_N} \xi_i^2} \sqrt{\sum_{i>1:\lambda_i \neq \lambda_N} \eta_i^2} + |\lambda_N| \sqrt{\sum_{i>1:\lambda_i = \lambda_N} \xi_i^2} \sqrt{\sum_{i>1:\lambda_i = \lambda_N} \eta_i^2} \\
&= \mu DE + |\lambda_N| \sqrt{\alpha - \alpha^2 - D^2} \sqrt{\beta - \beta^2 - E^2}
\end{aligned}$$

where $\mu = \max_{i>1:\lambda_i \neq \lambda_N} |\lambda_i|$. Note that the derangement graph Γ has $\mu \leq O((n-2)!)$. Hence, applying the above result to a cross-intersecting pair $\mathcal{A}, \mathcal{B} \subset S_n$ with $\sqrt{|\mathcal{A}||\mathcal{B}|} = (1-\delta)(n-1)!$, we obtain

$$\sqrt{1-\alpha-D^2/\alpha}\sqrt{1-\beta-E^2/\beta} \geq \frac{d_n\sqrt{\alpha\beta} - \mu(D/\sqrt{\alpha})(E/\sqrt{\beta})}{|\lambda_N|} \geq 1-\delta-O(1/n)$$

and therefore $1-\alpha-D^2/\alpha \geq (1-\delta)^2 - O(1/n)$, so $D^2 \leq \alpha(2\delta-\delta^2+O(1/n))$. Replacing δ with $2\delta - \delta^2 + O(1/n)$ in the proof of Theorem 3.1, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A}_{i \mapsto j}| \geq \frac{(n-1)!}{4\sqrt{2(n-1)\log n}}(1 - \sqrt{2\delta - \delta^2} - O(1/\sqrt{n})) = \omega((n-2)!)$$

since $\delta < 1 - c$. For each $k \neq j$, the pair $\mathcal{A}_{i \mapsto j}, \mathcal{B}_{i \mapsto k}$ is cross-intersecting, so as in Lemma 3.3, we have:

$$|\mathcal{A}_{i \mapsto j}||\mathcal{B}_{i \mapsto k}| \leq ((n-2)!)^2$$

Hence, for all $k \neq j$,

$$|\mathcal{B}_{i \mapsto k}| \leq o((n-2)!)$$

so summing over all $j \neq k$ gives

$$|\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq o((n-1)!)$$

Since $|\mathcal{B}| \geq |\mathcal{A}|$, $|\mathcal{B}| \geq c(n-1)!$, and therefore

$$|\mathcal{B}_{i \mapsto j}| \geq (c - o(1))(n-1)!$$

For each $k \neq j$, the pair $\mathcal{A}_{i \mapsto k}, \mathcal{B}_{i \mapsto j}$ is cross-intersecting, so as before, we have:

$$|\mathcal{A}_{i \mapsto k}||\mathcal{B}_{i \mapsto j}| \leq ((n-2)!)^2$$

Hence, for all $k \neq j$,

$$|\mathcal{A}_{i \mapsto k}| \leq O((n-3)!)$$

so summing over all $j \neq k$ gives

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}| \leq O((n-2)!)$$

Also, $|\mathcal{B}| = |\mathcal{B}_{i \mapsto j}| + |\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq (1+o(1))(n-1)!$, so $|\mathcal{A}| \geq c^2(1-o(1))(n-1)!$. Hence,

$$|\mathcal{A}_{i \mapsto j}| \geq c^2(1-o(1))(n-1)!$$

so by the same argument as above,

$$|\mathcal{B}_{i \mapsto k}| \leq O((n-3)!)$$

for all $k \neq j$, and therefore

$$|\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq O((n-2)!)$$

as well, proving Theorem 3.6. $\square$

We may use Theorem 3.6 to deduce two Hilton-Milner type results for cross-intersecting families:

Theorem 3.7. *For n sufficiently large, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families which are not both contained within the same 1-coset, then* $\min(|\mathcal{A}|, |\mathcal{B}|) \leq |\mathcal{C}| = (n-1)! - d_{n-1} - d_{n-2} + 1$, *with equality iff*

$$\begin{aligned} \mathcal{A} &= \{\sigma \in S_n : \sigma(i) = j,\ \sigma \text{ intersects } \tau\} \cup \{\rho\} \\ \mathcal{B} &= \{\sigma \in S_n : \sigma(i) = j, \sigma \text{ intersects } \rho\} \cup \{\tau\} \end{aligned}$$

for some $i, j \in [n]$ and some $\tau, \rho \in S_n$ which intersect and do not map i to j.

Proof. Suppose $\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|$. Applying Theorem 3.6 with any $c < 1 - 1/e$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i \mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i \mapsto j}| \leq O((n-2)!)$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1 \mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| \leq O((n-2)!)$$

Assume $\mathcal{A}$ is not contained within the 1-coset $\{\sigma \in S_n : \sigma(1) = 1\}$; let ρ be a permutation in $\mathcal{A}$ not fixing 1. Suppose for a contradiction that $\mathcal{A}$ contains another permutation π not fixing 1. As in the proof of Theorem 3.5, this implies that

$$|\mathcal{B}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}$$

and so by assumption,

$$|\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}| \geq d_{n-2} + 1$$

so $\mathcal{B} \setminus \mathcal{B}_{1 \mapsto 1}$ cannot be $(\log n)$-intersecting. As in the proof of Theorem 3.5, this implies that

$$|\mathcal{A}_{1 \mapsto 1}| \leq ((1 - 1/e)^2 + o(1))(n-1)!$$

giving

$$|\mathcal{A}| \leq ((1 - 1/e)^2 + o(1))(n-1)! < |\mathcal{C}|$$

—a contradiction. Hence,

$$\mathcal{A} = \mathcal{A}_{1 \mapsto 1} \cup \{\rho\}$$

If $\mathcal{B}$ were centred, then every permutation in $\mathcal{B}$ would have to fix 1 and intersect ρ, and we would have $|\mathcal{B}| = |\mathcal{B}_{1 \mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} < |\mathcal{C}|$, a

contradiction. Hence, $\mathcal{B}$ is also non-centred. Repeating the above argument with $\mathcal{B}$ in place of $\mathcal{A}$, we see that $\mathcal{B}$ contains just one permutation not fixing 1, τ say. Hence,

$$\mathcal{B} = \mathcal{B}_{1\mapsto 1} \cup \{\tau\}$$

Since $\min(|\mathcal{A}|, |\mathcal{B}|) \geq |\mathcal{C}|$, we have

$$\begin{aligned}
\mathcal{A}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \tau\} \\
\mathcal{B}_{1\mapsto 1} &= \{\sigma \in S_n : \sigma(1) = 1,\ \sigma \text{ intersects } \rho\}
\end{aligned}$$

proving the theorem. □

Similarly, we may prove

Theorem 3.8. *For n sufficiently large, if $\mathcal{A}, \mathcal{B} \subset S_n$ is a cross-intersecting pair of families which are not both contained within the same 1-coset, then*

$$|\mathcal{A}||\mathcal{B}| \leq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)$$

with equality iff

$$\mathcal{A} = \{\sigma \in S_n : \sigma(i) = j,\ \sigma \textit{ intersects } \rho\}, \quad \mathcal{B} = \{\sigma \in S_n : \sigma(i) = j\} \cup \{\rho\}$$

for some $i, j \in [n]$ and some $\rho \in S_n$ with $\rho(i) \neq j$.

Proof. Let $\mathcal{A}, \mathcal{B}$ be a cross-intersecting pair of families, not both centred, with $|\mathcal{A}||\mathcal{B}| \geq ((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)$. We have

$$\sqrt{|\mathcal{A}||\mathcal{B}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)!$$

so applying Theorem 3.6 with any $c < \sqrt{1 - 1/e}$, we see that there exist $i, j \in [n]$ such that

$$|\mathcal{A} \setminus \mathcal{A}_{i\mapsto j}|, |\mathcal{B} \setminus \mathcal{B}_{i\mapsto j}| \leq O((n-2)!)$$

By double translation, we may assume that $i = j = 1$, so

$$|\mathcal{A} \setminus \mathcal{A}_{1\mapsto 1}|, |\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| \leq O((n-2)!)$$

Therefore,

$$\sqrt{|\mathcal{A}_{1\mapsto 1}||\mathcal{B}_{1\mapsto 1}|} \geq (\sqrt{1 - 1/e} - O(1/n))(n-1)! \tag{14}$$

If $\mathcal{B}$ contains some permutation ρ not fixing 1, then

$$\mathcal{A}_{1\mapsto 1} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\}$$

and therefore

$$|\mathcal{A}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)!$$

Similarly, if $\mathcal{A}$ contains a permutation not fixing 1, then

$$|\mathcal{B}_{1\mapsto 1}| \leq (1 - 1/e + o(1))(n-1)!$$

By (14), both statements cannot hold (provided n is large), so we may assume that every permutation in $\mathcal{A}$ fixes 1, and that $\mathcal{B}$ contains some permutation ρ not fixing 1. Hence,

$$\mathcal{A} \subset \{\sigma \in S_n : \sigma(1) = 1, \sigma \text{ intersects } \rho\}$$

and

$$|\mathcal{A}| \leq (n-1)! - d_{n-1} - d_{n-2} = (1 - 1/e + o(1))(n-1)! \tag{15}$$

So by assumption,

$$|\mathcal{B}| \geq (n-1)! + 1 \tag{16}$$

Suppose for a contradiction that $\mathcal{B}$ contains another permutation $\pi \neq \rho$ such that $\pi(1) \neq 1$. Then, by the same argument as in the proof of Theorem 3.5, we would have

$$|\mathcal{A}| = |\mathcal{A}_{1\mapsto 1}| \leq (n-1)! - d_{n-1} - 2d_{n-2}$$

so by assumption,

$$|\mathcal{B}| \geq \frac{((n-1)! - d_{n-1} - d_{n-2})((n-1)! + 1)}{(n-1)! - d_{n-1} - 2d_{n-2}} = (n-1)! + \Omega((n-2)!)$$

This implies that $|\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}| = \Omega((n-2)!)$, so $\mathcal{B} \setminus \mathcal{B}_{1\mapsto 1}$ cannot be $(\log n)$-intersecting. Hence, by the same argument as in the proof of Theorem 3.5,

$$|\mathcal{A}_{1\mapsto 1}| \leq ((1 - 1/e)^2 + o(1))(n-1)!$$

Therefore,

$$\sqrt{|\mathcal{A}_{1\mapsto 1}||\mathcal{B}_{1\mapsto 1}|} \leq (1 - 1/e + o(1))(n-1)!$$

— contradicting (14). Hence, ρ is the only permutation in $\mathcal{B}$ not fixing 1, i.e.

$$\mathcal{B} = \mathcal{B}_{1\mapsto 1} \cup \{\rho\}$$

So we must have equality in (16), i.e.

$$\mathcal{B}_{1\mapsto 1} = \{\sigma \in S_n : \ \sigma(1) = 1\}$$

But then we must also have equality in (15), i.e.

$$\mathcal{A} = \{\sigma \in S_n : \ \sigma(1) = 1, \ \sigma \text{ intersects } \rho\}$$

proving the theorem. $\square$

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